

# Consequence of total lepton number violation in strongly magnetized iron white dwarfs

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## Abstract

The influence of neutrinoless electron to positron conversion on cooling of strongly magnetized iron white dwarfs is studied.

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## I. INTRODUCTION

As it was emphasized by Isern *et al.* [1, 2], due to simple cooling process in white dwarfs and precise measurements of luminosity curve became possible to use white dwarfs as a laboratory for analyzing problems of elementary particles physics. In the above works, it has particularly been suggested to study possible existence of axions on the basis of white dwarfs luminosity function. Following such an idea, we analyze the influence of lepton number violation on the luminosity of strongly magnetized iron white dwarfs. As is well known, the result of existence of Majorana type neutrino would be the lepton number violating process of electron capture by a nucleus  $X(A, Z)$

$$e^- + X(A, Z) \rightarrow X(A, Z - 2) + e^+, \quad (1.1)$$

which is an analogue of the neutrinoless double beta-decay, that is at present intensively studied [3].

Detailed study of strongly magnetized cold electron gas and its application to the magnetized white dwarfs has recently been done in Refs. [4, 5]. This theory stems from Landau quantization of the motion of electrons in magnetic field [6, 7] and of its modification to the case of very strong magnetic field [8]. It turns out that in systems with small number of Landau levels, which is restricted by a suitable choice of the strength of the magnetic field and of the maximum of the Fermi energy  $E_F$  of the electron gas, the mass of strongly magnetized white dwarf can be in the range  $(2.3-2.6) M_\odot$ , where  $M_\odot \approx 2 \times 10^{33}$  g is the solar mass. It means that the strong magnetic field can enhance the energy of the electron gas to such a level that its pressure can force the gravity to allow the white dwarf to possess more mass than it is restricted by the Chandrasekhar-Landau limit of  $1.44 M_\odot$  [9, 10].

Here we apply the above mentioned theory to study the influence of the double charge exchange reaction (1.1) on cooling of the strongly magnetized iron white dwarfs. The threshold for this reaction with the initial nucleus  $^{56}_{26}\text{Fe}$  and the final one,  $^{56}_{24}\text{Cr}$ , is  $\Delta=6.33$  MeV. So for the strongly magnetized iron white dwarfs with  $E_F \geq \Delta$  this reaction can take place. We shall consider  $\epsilon_F=E_F/m_e c^2=20, 46, 90$  and choose the strength of the magnetic field so that the value of the related parameter  $\gamma$  will allow us to restrict ourselves to one Landau level.

In Section II, we present methods and necessary input, needed for the calculations of the energy production due to the reaction (1.1), including the double charge exchange width per one elementary reaction for chosen values of  $\epsilon_F$  and  $\gamma$ . Then in Section III, we give our estimate of the energy production and in Section IV, we discuss our results and present our conclusions. At last, we provide in Appendix A invariant functions, entering the positron-electron annihilation probability. Our main conclusion is that for sufficiently strongly magnetized iron white dwarfs, the energy released in reaction (1.1) can effectively retard their cooling.

## II. METHODS AND INPUTS

In this section, we first discuss the necessary ingredients of the theory of strongly magnetized white dwarfs and then we present the formalism for the calculations of the double charge

exchange reaction (1.1).

### A. Theory of strongly magnetized white dwarfs

Theory of the strongly magnetized white dwarfs is based on Landau quantization of the motion of free electrons in a magnetic field  $\mathcal{B}$  directed along the z-axis [6, 7]. It is in detail discussed in Refs. [4, 5, 8]. In the relativistic case, one solves the Dirac equation, obtaining for the electron energy

$$E_\nu = m_e c^2 \left[ 1 + \left( \frac{p_z}{m_e c} \right)^2 + 2\nu \frac{e\hbar\mathcal{B}}{m_e^2 c^3} \right]^{1/2}. \quad (2.1)$$

Here  $m_e$  (e) is the electron mass (charge),  $c$  is the light velocity,  $p_z$  is the electron momentum along the z-axis,  $\nu = l + 1/2 + \sigma$  labels the Landau levels with the principal number  $l$ ,  $\sigma = \pm 1/2$ , and  $\hbar$  is Planck's constant. The ground level ( $\nu=0$ ) is obtained for  $l=0$  and  $\sigma=-1/2$ , and it has degeneracy 1. Other Landau levels possess degeneracy 2.

The last term at the r.h.s. of Eq. (2.1) can be written as

$$2\nu \frac{e\hbar\mathcal{B}}{m_e^2 c^3} = 2\nu \frac{\hbar\omega_H}{m_e c^2} = 2\nu \frac{\mathcal{B}}{\mathcal{B}_c}. \quad (2.2)$$

Here the cyclotron frequency  $\omega_H = \frac{e\mathcal{B}}{m_e c}$  and a critical magnetic field strength  $\mathcal{B}_c = \frac{m_e^2 c^3}{\hbar e} = 4.414 \times 10^{13}$  G. It is seen from Eqs. (2.1) and (2.2) that the electron becomes relativistic if  $\hbar\omega_H \geq m_e c^2$ , or if  $\mathcal{B} \geq \mathcal{B}_c$ .

In contrast to the density of electron states in the absence of the strong magnetic field, given as  $\frac{2}{(2\pi)^3 \hbar^3} d^3 p$ , the presence of such magnetic field modifies the number of electron states for a given level  $\nu$  to  $\frac{2g_\nu e\mathcal{B}}{(2\pi)^2 \hbar^2 c} dp_z$ . Then the sum over the electron states in the presence of the strong magnetic field is given by

$$\sum_e \rightarrow \sum_\nu \frac{2e\mathcal{B}}{(2\pi)^2 \hbar^2 c} g_\nu \int dp_z = \frac{2\gamma}{(2\pi)^2 \lambda_e^3} \sum_\nu g_\nu \int d\left(\frac{p_z}{m_e c}\right). \quad (2.3)$$

Here  $\gamma = \mathcal{B}/\mathcal{B}_c$ ,  $\lambda_e = \hbar/m_e c$  is the electron Compton wavelength, and  $g_\nu = (2 - \delta_{0,\nu})$  reflects the degeneracy of the Landau levels.

The equation between the Fermi energy  $\epsilon_F$  and the Fermi momentum  $x_F(\nu) = p_F(\nu)/m_e c$  for the Landau level, specified by  $\nu$  is obtained directly from Eq. (2.1)

$$\epsilon_F^2 = x_F^2(\nu) + (1 + 2\nu\gamma). \quad (2.4)$$

Then one obtains the electron number density from Eq. (2.3) as

$$n_{e^-} = \frac{2\gamma}{(2\pi)^2 \lambda_e^3} \sum_0^{\nu_{max}} g_\nu x_F(\nu). \quad (2.5)$$

TABLE I: The values of the Fermi energy  $\epsilon_F = E_F/m_e c^2$ , used in the present study. Further  $n_{e-}$  is the electron number density,  $\rho_{e-}$  is the corresponding electron density,  $\rho_m$  is the matter density, calculated according to Eq. (2.7) for the nuclei  $^{56}_{26}\text{Fe}$ . The values of  $\gamma$  are the smallest values, satisfying Eq. (2.6) and the mean energy  $\bar{\epsilon}_{e-}$  is defined in Eq. (2.9).

$\epsilon_F$	$n_{e-}/10^{33} \text{ [1/cm}^3\text{]}$	$\rho_{e-}/10^6 \text{ [g/cm}^3\text{]}$	$\rho_m/10^{10} \text{ [g/cm}^3\text{]}$	$2\gamma$	$\bar{\epsilon}_{e-} \text{ [MeV]}$
20	3.52	3.20	1.26	400	5.1
46	42.8	13.4	15.2	2116	11.75
90	321	293	114	8100	23

The values of  $\nu$  in the sum are restricted by the condition that in Eq. (2.5)  $x_F(\nu) \geq 0$ , which provides that  $\epsilon_F^2 - (1 + 2\nu\gamma) \geq 0$ , from which the inequality follows

$$\nu_{max} \leq \text{integer} \left( \frac{\epsilon_{Fmax}^2 - 1}{2\gamma} \right). \quad (2.6)$$

Supposing that the white dwarfs are electrically neutral, one deduces for the system of the electron gas and one sort of nuclei the matter density

$$\rho_m = \mu_{e-} m_U n_{e-} = \frac{n_{e-}}{Z} m_A, \quad (2.7)$$

where  $\mu_{e-} = A/Z$  is the molecular weight per electron [ $A(Z)$  is the mass (atomic) number of the nucleus],  $m_U$  is the atomic mass unit and  $m_A$  is the mass of the nucleus with the mass number  $A$ . For the lightest nuclei,  $\mu_{e-} = 2$ , but, e.g., for  $^{56}_{26}\text{Fe}$  one obtains  $\mu_{e-} = 2.15$ .

We also present the electron energy density at zero temperature

$$\begin{aligned} \varepsilon_{e-} &= \frac{2\gamma}{(2\pi)^2 \lambda_e^3} \sum_0^{\nu_{max}} g_\nu \int_0^{x_F} E_\nu dx(\nu) \\ &= \frac{2\gamma m_e c^2}{(2\pi)^2 \lambda_e^3} \sum_0^{\nu_{max}} g_\nu \int_0^{x_F} [1 + x^2(\nu) + 2\nu\gamma]^{1/2} dx(\nu) \\ &= \frac{\gamma m_e c^2}{(2\pi)^2 \lambda_e^3} \sum_0^{\nu_{max}} g_\nu \left[ x_F (\epsilon_F^2 + 2\nu\gamma)^{1/2} + (1 + 2\nu\gamma) \ln \frac{x_F + (\epsilon_F^2 + 2\nu\gamma)^{1/2}}{(1 + 2\nu\gamma)^{1/2}} \right]. \end{aligned} \quad (2.8)$$

The energy per electron is then

$$\bar{\epsilon}_{e-} = \varepsilon_{e-} / n_{e-}. \quad (2.9)$$

Numerically one gets  $\bar{\epsilon}_{e-} = E_F/2$ .

Here we restrict ourselves to the strongly magnetized iron white dwarfs possessing the ground Landau level ( $\nu_{max} = 0$ ) and consider  $\epsilon_F = 20, 46$  and  $90$ . Our choice of the parameter  $\gamma$  is  $\gamma = \epsilon_F^2/2$ , which is the minimal value of  $\gamma$ , satisfying Eq. (2.6). In Table I, we present, together with the values of  $\epsilon_F$ , the related quantities, needed for the calculations.

## B. Reaction rate

The process of  $(e^-, e^+)$  conversion (1.1) is very similar to the neutrinoless double beta decay ( $0\nu\beta\beta$ -decay)

$$X(A, Z) \rightarrow X(A, Z + 2) + e^- + e^-, \quad (2.10)$$

Both processes violate total lepton number by two units and therefore, take place if and only if neutrinos are Majorana particles with the non-zero mass. In addition, as we shall show below, the  $(e^-, e^+)$  conversion rate is, like the  $0\nu\beta\beta$ -decay rate, proportional to the squared absolute value of the effective mass of Majorana neutrinos  $|\langle m_\nu \rangle|^2$ . This quantity takes the form

$$m_\nu = \sum_{i=1}^3 U_{ei}^2 m_i. \quad (2.11)$$

Here  $U$  is the  $3 \times 3$  Pontecorvo-Maki-Nakagawa-Sakata unitary mixing matrix and  $m_i$  ( $i = 1, 2, 3$ ) is the mass of the  $i$ -th light neutrino. Let us note that  $m_\nu$  depends on neutrino oscillation parameters  $\theta_{12}$ ,  $\theta_{13}$ ,  $\Delta m_{\text{SUN}}^2$ ,  $\Delta m_{\text{ATM}}^2$ , the lightest neutrino mass and the type of the neutrino mass spectrum (normal or inverted).

From the most precise experiments on the search for the  $0\nu\beta\beta$ -decay [11–13], by use of nuclear matrix elements (NME) of Ref. [14], the following stringent bounds were inferred [3]

$$\begin{aligned} |\langle m_\nu \rangle| &< (0.20 - 0.32) \text{ eV} \quad ({}^{76}\text{Ge}), \\ &< (0.33 - 0.46) \text{ eV} \quad ({}^{130}\text{Te}), \\ &< (0.17 - 0.30) \text{ eV} \quad ({}^{136}\text{Xe}). \end{aligned} \quad (2.12)$$

However, there exists a claim of the observation of the  $0\nu\beta\beta$ -decay of  ${}^{76}\text{Ge}$ , made by some participants of the Heidelberg-Moscow collaboration [15]. Their estimated value of the effective Majorana mass (assuming a specific value for the NME) is  $|\langle m_\nu \rangle| \simeq 0.4 \text{ eV}$ . In future experiments, CUORE ( ${}^{130}\text{Te}$ ), EXO, KamLAND-Zen ( ${}^{136}\text{Xe}$ ), MAJORANA/GERDA ( ${}^{76}\text{Ge}$ ), SuperNEMO ( ${}^{82}\text{Se}$ ), SNO+ ( ${}^{150}\text{Nd}$ ), and others [16], a sensitivity

$$|\langle m_\nu \rangle| \simeq \text{a few } 10^{-2} \text{ eV} \quad (2.13)$$

is planned to be reached, which is the region of the inverted hierarchy of neutrino masses. In the case of the normal mass hierarchy,  $|\langle m_\nu \rangle|$  is too small in order to be probed in the  $0\nu\beta\beta$ -decay experiments of the next generation.

For the sake of simplicity, the  $(e^-, e^+)$  conversion on nuclei is considered only for the ground state to ground state transition, which is assumed to give the dominant contribution to the  $(e^-, e^+)$  conversion rate. The spin and parity of initial ( ${}^{56}\text{Fe}$ ) and final ( ${}^{56}\text{Cr}$ ) nuclei in the ground state are equal, namely  $0^+$ . The incoming electron and outgoing positron are considered to be presumably in the  $s_{1/2}$  wave-states

$$\psi_{e^-}^{(s_{1/2})}(P_{e^-}) \approx \sqrt{F_0(Z, E_{e^-})} u(P_{e^-}), \quad (2.14)$$

$$\psi_{e^+}^{(s_{1/2})}(P_{e^+}) \approx \sqrt{F_0(Z - 2, E_{e^+})} u(P_{e^+}). \quad (2.15)$$

The Coulomb interaction of electron and positron with the nucleus is taken by the relativistic Fermi functions  $F(Z, E_{e-})$  and  $F(Z - 2, E_{e+})$  [17], respectively. Normalization of spinors is  $u^\dagger(P)u(P) = 1$  and  $v^\dagger(P)v(P) = 1$ .  $P_{e-} \equiv (E_{e-}, \mathbf{p}_{e-})$  and  $P_{e+} \equiv (E_{e+}, \mathbf{p}_{e+})$  are the 4-momenta of electron and positron, respectively. The above approximation is expected to work well for energy of incoming electron below 30, or even 50 MeV.

The leading order ( $e^-$ ,  $e^+$ ) conversion matrix element reads

$$\langle f|S^{(2)}|i\rangle = 2\pi\delta(E_{e+} - E_{e-} + E_f - E_i)\langle f|T^{(2)}|i\rangle, \quad (2.16)$$

with

$$\begin{aligned} \langle f|T^{(2)}|i\rangle &= i \langle m_\nu \rangle^* \frac{1}{4\pi} G_\beta^2 \sqrt{F_0(Z, E_{e-})} \sqrt{F_0(Z - 2, E_{e+})} \bar{v}(P_{e+})(1 + \gamma_5)u(P_{e-}) \times \\ &\quad \frac{g_A^2}{R} M^{(e\beta+)}. \end{aligned} \quad (2.17)$$

Here  $G_\beta = G_F \cos \theta_c$  and  $E_i$  ( $E_f$ ) is the energy of the initial (final) nuclear ground state. Later on, we neglect the kinetic energy of the final nucleus. The conventional normalization factor of NME,  $M^{(e\beta+)}$ , involves the nuclear radius  $R = 1.2 A^{1/3}$  fm. For the weak axial coupling constant  $g_A$ , we adopt the value  $g_A = 1.269$ .

The NME in Eq. (2.16), defined as

$$M^{(e\beta+)} = -\frac{M_F^{(e\beta+)}}{g_A^2} + M_{GT}^{(e\beta+)}, \quad (2.18)$$

contains the Fermi,  $M_F^{(e\beta+)}$ , and Gamow-Teller,  $M_{GT}^{(e\beta+)}$ , contributions. They take the following form

$$\begin{aligned} M_F^{(e\beta+)} &= \frac{4\pi R}{(2\pi)^3} \int \frac{d\mathbf{q}}{2q} f_V^2(q^2) \times \\ &\quad \sum_n \left( \frac{\langle 0_f^+ | \sum_l \tau_l^+ e^{-i\mathbf{q}\cdot\mathbf{r}_l} | n \rangle \langle n | \sum_m \tau_m^+ e^{i\mathbf{p}_{e-}\cdot\mathbf{r}_m} e^{i\mathbf{q}\cdot\mathbf{r}_m} | 0_i^+ \rangle}{q - E_{e-} + E_n - E_i + i\varepsilon_n} \right. \\ &\quad \left. + \frac{\langle 0_f^+ | \sum_m \tau_m^+ e^{i\mathbf{q}\cdot\mathbf{r}_m} | n \rangle \langle n | \sum_l \tau_l^+ e^{-i\mathbf{p}_{e+}\cdot\mathbf{r}_l} e^{-i\mathbf{q}\cdot\mathbf{r}_l} | 0_i^+ \rangle}{q + E_{e+} + E_n - E_i + i\varepsilon_n} \right), \end{aligned} \quad (2.19)$$

$$\begin{aligned} M_{GT}^{(e\beta+)} &= \frac{4\pi R}{(2\pi)^3} \int \frac{d\mathbf{q}}{2q} f_A^2(q^2) \times \\ &\quad \sum_n \left( \frac{\langle 0_f^+ | \sum_l \tau_l^+ \boldsymbol{\sigma}_l e^{-i\mathbf{q}\cdot\mathbf{r}_l} | n \rangle \cdot \langle n | \sum_m \tau_m^+ \boldsymbol{\sigma}_m e^{i\mathbf{p}_{e-}\cdot\mathbf{r}_m} e^{i\mathbf{q}\cdot\mathbf{r}_m} | 0_i^+ \rangle}{q - E_{e-} + E_n - E_i + i\varepsilon_n} \right. \\ &\quad \left. + \frac{\langle 0_f^+ | \sum_m \tau_m^+ \boldsymbol{\sigma}_m e^{i\mathbf{q}\cdot\mathbf{r}_m} | n \rangle \cdot \langle n | \sum_l \tau_l^+ \boldsymbol{\sigma}_l e^{-i\mathbf{p}_{e+}\cdot\mathbf{r}_l} e^{-i\mathbf{q}\cdot\mathbf{r}_l} | 0_i^+ \rangle}{q + E_{e+} + E_n - E_i + i\varepsilon_n} \right). \end{aligned} \quad (2.20)$$

We use the conventional dipole parametrization for the nucleon form factors, normalized to unity

$$f_V(q^2) = \left(1 + \frac{q^2}{M_V^2}\right)^{-2}, \quad f_A(q^2) = \left(1 + \frac{q^2}{M_A^2}\right)^{-2}, \quad (2.21)$$

with  $M_V = 0.71$  GeV,  $M_A = 1.091$  GeV. In the denominators of Eqs. (2.19) and (2.20),  $E_n$  and  $\varepsilon_n$  are the energy and width of the  $n$ -th intermediate nuclear state, respectively.

For the considered  $(e^-, e^+)$  conversion on  $^{56}\text{Fe}$ , the typical momentum of intermediate neutrinos is about 200 MeV (like in the case of the  $0\nu\beta\beta$ -decay [18]), i.e., significantly larger in comparison with typical excitation energies of the intermediate nuclear states. Thus, in Eqs. (2.19) and (2.20), we complete the sum over the virtual intermediate nuclear states by closure, after replacing  $E_n - E_i$  and  $\varepsilon_n$  with some average values  $\langle E_n - E_i \rangle$  and  $\varepsilon$ , respectively

$$\begin{aligned} \sum_n \frac{|n\rangle\langle n|}{q - E_{e^-} + E_n - E_i + i\varepsilon_n} &\approx \frac{1}{q - E_{e^-} + \langle E_n - E_i \rangle + i\varepsilon}, \\ \sum_n \frac{|n\rangle\langle n|}{q + E_{e^+} + E_n - E_i + i\varepsilon_n} &\approx \frac{1}{q + E_{e^+} + \langle E_n - E_i \rangle + i\varepsilon}. \end{aligned} \quad (2.22)$$

As a result, the nuclear matrix element  $M_\nu^{(e^-e^+)}$ , decomposed into the contributions coming from direct and cross Feynman diagrams, takes the form

$$M^{(e\beta^+)} = M_{\text{dir.}}^{(e\beta^+)} + M_{\text{cro.}}^{(e\beta^+)}, \quad (2.23)$$

with

$$M_{\text{dir.}}^{(e\beta^+)} = \langle 0_i^+ | \sum_{lm} \tau_l^+ \tau_m^+ \frac{R}{\pi} \int_0^\infty \frac{j_0(qr_{lm}) f^2(q^2) q dq}{q - E_{e^-} + \langle E_n - E_i \rangle + i\varepsilon} \left( \boldsymbol{\sigma}_l \cdot \boldsymbol{\sigma}_m - \frac{1}{g_A^2} \right) | 0_f^+ \rangle, \quad (2.24)$$

$$M_{\text{cro.}}^{(e\beta^+)} = \langle 0_i^+ | \sum_{lm} \tau_l^+ \tau_m^+ \frac{R}{\pi} \int_0^\infty \frac{j_0(qr_{lm}) f^2(q^2) q dq}{q + E_{e^+} + \langle E_n - E_i \rangle + i\varepsilon} \left( \boldsymbol{\sigma}_l \cdot \boldsymbol{\sigma}_m - \frac{1}{g_A^2} \right) | 0_f^+ \rangle. \quad (2.25)$$

It is important to note that the value of  $E_r \equiv -E_{e^-} + \langle E_n - E_i \rangle$  is negative for considered values of  $E_{e^-}$  and the studied nuclear system  $A = 56$ . Therefore, the contribution of direct Feynman diagram with the light intermediate neutrino has the pole at  $q = -E_r - i\varepsilon$ , as it follows from Eq. (2.24). As a consequence, there is a non-zero imaginary part of the  $(e^-, e^+)$  conversion amplitude, which should be considered as well.

The following comment is in order. In the expression (2.19) for nuclear matrix elements  $M^{(e\beta^+)}$ , we neglected the contributions of the higher order terms of the nucleon current (weak-magnetism, induced pseudoscalar coupling). As suggested by the analogy with  $0\nu\beta\beta$ -decay, these terms should not be essential for the light neutrino exchange mechanism.

Now we are ready to write down the expression for  $g.s. \rightarrow g.s.$   $(e^-, e^+)$  conversion rate. The differential capture rate can be written as

$$d\Gamma^{(e\beta^+)} = \sum | \langle f | T | i \rangle |^2 2\pi \delta(E_{e^-} + E_i - E_f - E_{e^+}) \frac{d\mathbf{p}_{e^+}}{(2\pi)^3} V, \quad (2.26)$$

where  $V$  is a volume of a phase factor. After the summed and squared T-matrix is calculated we get

$$\Gamma^{(e\beta^+)} = |\langle m_\nu \rangle|^2 \frac{1}{V} \frac{1}{16\pi^3} \left( \frac{G_\beta}{\sqrt{2}} \right)^4 F_0(Z, E_{e^-}) F_0(Z - 2, E_{e^+}) \frac{g_A^4}{R^2} \left| M^{(e\beta^+)} \right|^2 p_{e^+} E_{e^+}. \quad (2.27)$$

Here,  $E_{e^+} = E_{e^-} - \Delta$ .

The above result is for a single electron in a volume  $V$ . By assuming the density of electrons  $n_{e^-}$  (we replace  $1/V$  with  $n_{e^-}$ ), the reaction rate per nucleus is

$$\begin{aligned} \Gamma^{(e\beta^+)} &= m_e \frac{|\langle m_\nu \rangle|^2}{m_e^2} \frac{1}{16\pi^3} \left( \frac{G_\beta m_e^2}{\sqrt{2}} \right)^4 F_0(Z, E_{e^-}) F_0(Z-2, E_{e^+}) \\ &\times \frac{g_A^4}{(R^2 m_e^2)} \left| M^{(e\beta^+)} \right|^2 \frac{p_{e^+}}{m_e^2} \frac{E_{e^+}}{m_e^3} \frac{n_{e^-}}{m_e^3}. \end{aligned} \quad (2.28)$$

For the reaction (1.1), occurring in the strongly magnetized white dwarfs, one should sum up in Eq. (2.28) over the electron energies according to Eq. (2.3). This procedure provides the change

$$F_0(Z, E_{e^-}) F_0(Z-2, E_{e^+}) \frac{p_{e^+}}{m_e^2} \frac{E_{e^+}}{m_e^3} \frac{n_{e^-}}{m_e^3} \rightarrow \phi(\epsilon_F, \gamma). \quad (2.29)$$

The function  $\phi(\epsilon_F, \gamma)$  is defined as

$$\begin{aligned} \phi(\epsilon_F, \gamma) &= \frac{2\gamma}{(2\pi)^2 \lambda_e^3 m_e^3} \int_{Q+1}^{\epsilon_F} \left[ \frac{(\epsilon_{e^-} - Q)^2 - 1}{\epsilon_{e^-} - 1} \right]^{1/2} (\epsilon_{e^-} - Q) \epsilon_{e^-} \\ &\times F_0(Z, \epsilon_{e^-}) F_0(Z-2, \epsilon_{e^+}) d\epsilon_{e^-}, \end{aligned} \quad (2.30)$$

where  $\epsilon_{e^\pm} = E_{e^\pm}/m_e c^2$  and  $Q = \Delta/m_e c^2$ . The calculated values of  $\phi$  for chosen  $\epsilon_F$  and  $\gamma$  are

$$\begin{aligned} \phi(20, 200) &= 1.80 \times 10^3, \\ \phi(46, 1058) &= 7.94 \times 10^5, \\ \phi(90, 4050) &= 3.57 \times 10^7. \end{aligned} \quad (2.31)$$

Then the reaction rate reads

$$\Gamma^{(e\beta^+)} = m_e \frac{|\langle m_\nu \rangle|^2}{m_e^2} \frac{1}{16\pi^3} \left( \frac{G_\beta m_e^2}{\sqrt{2}} \right)^4 \frac{g_A^4}{(R^2 m_e^2)} \left| M^{(e\beta^+)} \right|^2 \phi(\epsilon_F, \gamma). \quad (2.32)$$

### C. Nuclear matrix element

We use the Quasiparticle Random Phase Approximation (QRPA) [14, 18] to calculate nuclear matrix element for the transition ( $e^-$ ,  $e^+$ ) on  $^{56}\text{Fe}$ . For the  $A=56$  system, the single-particle model space consisted of  $0 - 4\hbar\omega$  oscillator shells, both for the protons and neutrons. The single particle energies are obtained by using a Coulomb-corrected Woods-Saxon potential. We derive the two-body G-matrix elements from the Charge Dependent Bonn one-boson exchange potential [19] within the Brueckner theory. The pairing interactions are adjusted to fit the empirical pairing gaps.

The particle-particle and particle-hole channels of the G-matrix interaction of the nuclear Hamiltonian are renormalized by introducing the parameters  $g_{pp}$  and  $g_{ph}$ , respectively. The



calculations are carried out for  $g_{ph} = 1.0$ . The particle-particle strength parameter  $g_{pp}$  of the QRPA is fixed by the assumption that the matrix element  $M_{GT}^{\nu\bar{\nu}}$  of the lepton number conserving process of electron to positron conversion on nuclei with emission of neutrino and antineutrino

$$e^- + X(A, Z) \rightarrow X(A, Z - 2) + e^+ + \nu_e + \bar{\nu}_e, \quad (2.33)$$

is within the range  $(0, 0.30) \text{ MeV}^{-1}$ . Recall that a comparable quantity  $M_{GT}^{2\nu}$ , associated with the two-neutrino double beta decay of  $^{48}\text{Se}$ ,  $^{76}\text{Ge}$ ,  $^{82}\text{Se}$ ,  $^{128,130}\text{Te}$  and  $^{136}\text{Xe}$ , does not exceed the above range by assuming the weak-axial coupling constant  $g_A$  to be unquenched ( $g_A = 1.269$ ) or quenched ( $g_A = 1.0$ ).

As we already commented above, the matrix element  $M_{\text{dir.}}^{(e\beta^+)}$  (see Eq.(2.24)) of the direct contribution contains an imaginary part that stems from the pole of the integrand at  $q = -E_r - i\varepsilon$ . The averaged energy of the intermediate nuclear states  $\langle E_n - E_i \rangle$  is assumed to be 5 MeV. Taking into account that the widths of low-lying nuclear states are negligible in comparison to their energies, one can separate the imaginary and real parts of this matrix element using the well-known formula

$$\frac{1}{\alpha + i\varepsilon} = \mathcal{P}\frac{1}{\alpha} - i\pi\delta(\alpha), \quad (2.34)$$

valid in the limit  $\varepsilon \rightarrow 0$ .

In Fig. 1, the absolute value of the nuclear matrix element  $M^{(e\beta^+)}$  for  $^{56}\text{Fe}$  is plotted as function of energy of incoming electron  $E_{e^-}$ . The allowed region is due to the uncertainty, associated with fixing the particle-particle parameter  $g_{pp}$ . We found that the contribution from imaginary part of  $M^{(e\beta^+)}$  is small but increasing with  $E_{e^-}$  like the hole modulus of the  $(e^-, e^+)$  nuclear matrix element. For further qualitative analysis of the  $(e^-, e^+)$  capture rate we shall consider

$$|M^{(e\beta^+)}| \approx 3, \quad (2.35)$$

for  $E_{e^-}$  from the range  $(6.33, 50) \text{ MeV}$ .

### III. ENERGY PRODUCTION OF THE REACTION $e^- \rightarrow e^+$ IN THE STRONGLY MAGNETIZED IRON WHITE DWARFS

In order to estimate the energy production in one event,  $\bar{\varepsilon}_r$ , in the reaction (1.1), we calculated the two-photon positron-electron annihilation probability per unique volume within the quantum electrodynamics framework [20, 21] and integrated it over the energies of electrons  $\epsilon_f = E_f/m_e$  interacting with the positron in the final state of reaction (1.1), according to the prescription (2.3). As a result, we obtained

$$\bar{\Gamma}_i = \frac{2\gamma}{(2\pi)^2 \lambda_e^3} \int_1^{\epsilon_F} \frac{\epsilon_f}{(\epsilon_f^2 - 1)^{1/2}} \Gamma_i d\epsilon_f, \quad i = 0, 1. \quad (3.1)$$

In this section, we keep  $\hbar = c = 1$  and our notations follow closely those of Ref. [20], ch. 8. Further,

$$\Gamma_i = \frac{(2\pi)^4}{8} \sum_{\epsilon, \epsilon', \sigma, \sigma'} \int |M|^2 \delta^{(4)}(p_f + p_{e^+} - k - k') f_i(k^0 + k'^0) d\mathbf{k} d\mathbf{k}', \quad (3.2)$$

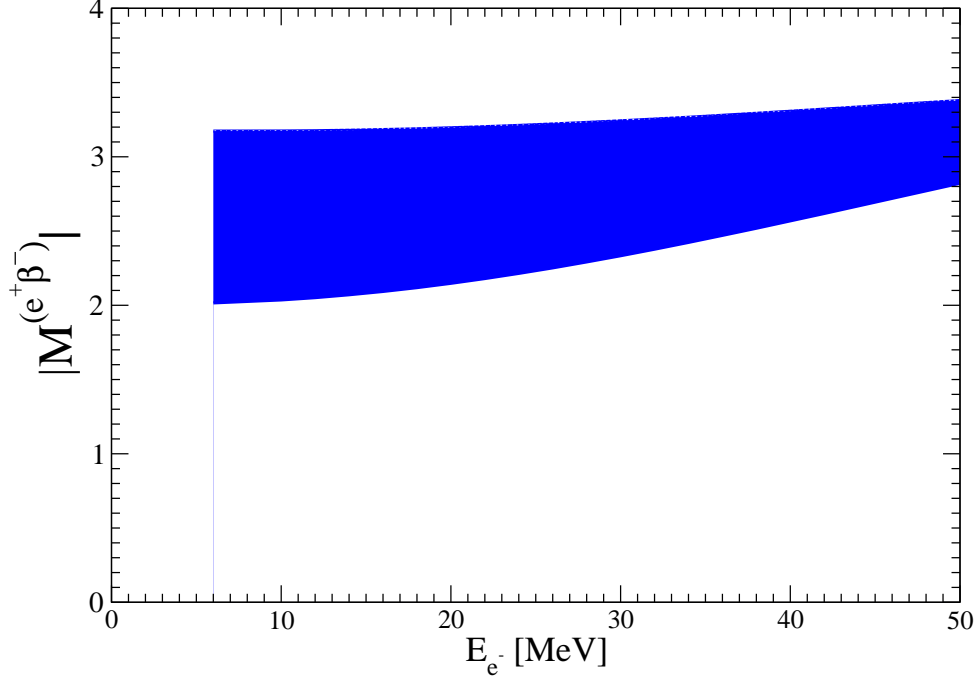


FIG. 1: (Color online) The absolute value of nuclear matrix element  $M^{(e\beta^+)}$  for  $^{56}\text{Fe}$  as function of energy of incoming electron  $E_{e-}$ . The allowed region of calculated nuclear matrix element is determined by fixing the particle-particle parameter to  $M_{GT}^{\nu\bar{\nu}}$  from the range  $(0, 0.30) \text{ MeV}^{-1}$ .

$f_0(k^0 + k'^0) = 1$ ,  $f_1(k^0 + k'^0) = k^0 + k'^0$ , the sum is performed over two photon linear polarizations  $\epsilon, \epsilon'$  and the average is done over the electron (positron) spin z-component  $\sigma$  ( $\sigma'$ ), the final photons have the 4-momenta  $k^\mu$  and  $k'^\mu$  ( $k^0 = |\mathbf{k}| = \omega$ ,  $k'^0 = |\mathbf{k}'| = \omega'$ ), the positron (electron) 4-momentum is  $p_{e+}^\mu$  ( $p_f^\mu$ ), also  $E_{e+} = p_{e+}^0$  and  $E_f = p_f^0$ . In its turn, the amplitude  $M$  is

$$M = -\frac{e^2}{4(2\pi)^3 \sqrt{(\omega \omega')}} \bar{v}(\mathbf{p}_{e+}, \sigma') \{ \not{\epsilon}' [-i(\not{p}_f - \not{k}) + m_e] \not{\epsilon} / (p_f \cdot k) + \not{\epsilon} [-i(\not{p}_f - \not{k}') + m_e] \not{\epsilon}' / (p_f \cdot k') \} u(\mathbf{p}_f, \sigma). \quad (3.3)$$

The calculation of the  $|M|^2$  reduces to calculations of traces. As a result we obtain

$$\Gamma_i = \frac{\alpha^2 \pi}{2m_e^2 \epsilon_{e+}} \int_{-1}^{+1} dx \frac{\bar{\omega}}{\epsilon_f + \epsilon_{e+} + (|\bar{\mathbf{p}}_{e+}| - |\bar{\mathbf{p}}_f|)x} \{ A_0 + A_1/(\bar{p}_f \cdot \bar{k})^2 + A_2/((\bar{p}_f \cdot \bar{k})(\bar{p}_f \cdot \bar{k}')) + A_3/(\bar{p}_f \cdot \bar{k}')^2 \} m_e^i f_i(\bar{\omega} + \bar{\omega}'). \quad (3.4)$$

Here the bared quantities are expressed in the units of the electron mass and

$$\omega = \frac{m_e^2 + E_{e+} E_f - |\mathbf{p}_{e+}| |\mathbf{p}_f|}{E_{e+} + E_f + (|\mathbf{p}_{e+}| - |\mathbf{p}_f|)x}, \quad \omega' = E_{e+} + E_f - \omega. \quad (3.5)$$

The scalar function  $A_0$  comes from the part of the traces that do not contain the factors  $(\epsilon \cdot p_f)$  and  $(\epsilon' \cdot p_f)$ . The functions  $A_i$ ,  $i = 0, 1, 2, 3$ , are presented in the Appendix A.

TABLE II: The values of the Fermi energy  $\epsilon_F = E_F/m_e$ , used in the present study. Further  $R_{WD}$  is the radius of the white dwarf,  $\Delta L$  ( $\Delta T$ ) is the calculated change in the luminosity (surface temperature), and the released energy in single reaction  $\bar{\epsilon}_r$  is defined in Eq. (3.6).

$\epsilon_F$	$R_{WD}/10^4$ [m]	$\Delta L$ [W/s]	$\Delta T$ [K]	$\bar{\epsilon}_r$ [MeV]
20	42.3	$2.7 \times 10^{11}$	27	9.1
46	18.6	$3.3 \times 10^{14}$	244	25.2
90	9.4	$2.9 \times 10^{17}$	1847	49.1

In the next step, we include  $\bar{\Gamma}_i$  into the function  $\phi(\epsilon_F, \gamma)$ , thus obtaining  $\phi_0(\epsilon_F, \gamma)$  and  $\phi_1(\epsilon_F, \gamma)$ . The energy produced in one event per one second is then calculated as

$$\bar{\epsilon}_r = \phi_1(\epsilon_F, \gamma) / \phi_0(\epsilon_F, \gamma). \quad (3.6)$$

In the interval of 1 year, the number of reactions in  $1 \text{ cm}^3$  is  $n_r = n_m \Gamma$ , where  $n_m$  is the number density of matter. Then the released energy in  $1 \text{ cm}^3$  per 1 year is

$$E = n_m \Gamma \bar{\epsilon}_r, [J \text{ cm}^{-3} \text{ y}^{-1}]. \quad (3.7)$$

Let us consider the white dwarf with the mass  $M_{WD} = 2 M_\odot \approx 4 \times 10^{33} \text{ g}$ . Its volume is  $V_{WD} = M_{WD} / \rho_m$ , from which one obtains the radius  $R_{WD}$ . In the full volume, the released energy per 1 year is  $EV$  [ $J \text{ y}^{-1}$ ], from which one obtains directly the change in luminosity (released energy per 1 second),  $\Delta L = EV / 3.154 \times 10^7$  [W/s]. The influence on the surface temperature of the white dwarf can be calculated from the equation

$$\Delta T = \left( \frac{\Delta L}{s 4\pi R_{WD}^2} \right)^{1/4}. \quad (3.8)$$

Here  $s = 5.67 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$ .

Next we present in Table II the calculated change in the luminosity and in the surface temperature of the strongly magnetized iron white dwarfs, using the necessary input from Table I,  $|\langle m_\nu \rangle| = 0.3 \text{ eV}$  and Eqs. (2.31), (2.32), (2.35). Besides,  $R = 1.2 \text{ A}^{1/3} \approx 4.59 \text{ fm}$ .

Our model of white dwarfs with the constant density is not fully realistic. On the other hand, more realistic model, considered in [4, 5] for  $\epsilon_F = 20$  shows (see Fig. 3b) that for the mass of the white dwarf  $M_{WD} \approx 2 M_\odot$  is  $R_{WD} \approx 8.8 \times 10^5 \text{ m}$ , which is about 2 times larger than in our model. So in calculating the change in the temperature according to Eq. (3.8), we used the radius twice as large as obtained from the model with the constant density. The correction to it for  $\epsilon_F = 46$  and  $\epsilon_F = 90$  can be estimated as follows. For  $\epsilon_F = 200$ , the value of the radius of the model [4, 5] is  $R_{WD} \approx 7 \times 10^4 \text{ m}$  [22], which is about 1.65 times larger than the constant density model  $R_{WD}$ . Applying the linear interpolation to get a corrected value of  $R_{WD}$  for  $\epsilon_F = 46$  ( $\epsilon_F = 90$ ), we obtained that the calculated values of  $\Delta T$  in Table II are underestimated by  $\sim 1 \%$  ( $4 \%$ ).

#### IV. DISCUSSION OF THE RESULTS AND CONCLUSIONS

In this work, we studied the influence of the double charge exchange reaction (1.1) on the cooling of the strongly magnetized iron white dwarfs. This reaction is closely related to the neutrinoless double beta decay process, which is nowadays intensively studied [3]. Both processes violate lepton number by two units and, therefore, take place if and only if neutrinos are Majorana particles with non-zero mass. For the case of light neutrino exchange, the  $(e^-, e^+)$  conversion and the  $0\nu\beta\beta$ -decay rates are proportional to the squared absolute value of the effective mass of Majorana neutrinos,  $|\langle m_\nu \rangle|^2$ .

As it is seen from our Table II, the double charge exchange reaction (1.1) can for sufficiently magnetized iron white dwarfs effectively retard their cooling by pumping the energy of the Fermi sea of electrons to the thermal energy of ions. It also follows that the above studied type of white dwarfs could, in principle, serve as a tool for observation of existence of the Majorana neutrino. The analysis of data on magnetic white dwarfs with  $\mathcal{B} \approx 1\text{--}2 \times 10^7$  G have very recently been published in Ref. [23]. As it can be seen from Table 4, error in provisional estimates of the temperature is at the level of 10 %. However, our estimates in change of the temperature due to the reaction (1.1) could be of interest for analysis of magnetized iron white dwarfs with  $\mathcal{B} > \mathcal{B}_c = 4.414 \times 10^{13}$  G.

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- [1] J. Isern, S. Catalan, E. Garcia-Berro, M. Lalaris, S. Torres, in Proc. of the 6th Patras Workshop on Axions, WIMPs and WISPs, ed. by L. Baudis, M. Schumann, Hamburg, DESY, 2010, p.77-80.
  - [2] J. Isern, E. Garcia-Berro, S. Torres, S. Catalan, *Astrophys. J.* **682** (2008) L109 .
  - [3] J.D. Vergados, H. Ejiri, F. Šimkovic, *Rep. Prog. Phys.* **75** (2012) 106301 .
  - [4] A. Kundu, B. Mukhopadhyay, *Mod. Phys. Lett.* **A27** (2012) 1250084 .
  - [5] U. Das, B. Mukhopadhyay, *Phys. Rev. D* **86** (2012) 042001 .
  - [6] L.D. Landau, E.M. Lifshits, *Quantum Mechanics*, State Publishing House of Physical and Mathematical Literature, Moscow, 1963.
  - [7] M. Strickland, V. Dexheimer, D.P. Menezes, arXiv:1209.3276 [nucl-th].
  - [8] D. Lai, S.I. Shapiro, *Astrophys. J.* **383** (1991) 745 .
  - [9] S. Chandrasekhar, *Astrophys. J.* **74** (1931) 81 .
  - [10] L.D. Landau, *Phys. Z. Sowjetunion* **1** (1932) 285 .
  - [11] L. Baudis et al., *Phys. Rev. Lett.* **83** (1999) 41 .
  - [12] C. Arnaboldi et al. (CUORE Collaboration), *Phys. Lett.* **B584** (2004) 260 .

- [13] A. Gando *et al.* (KamLAND-Zen Collaboration), *Phys. Rev. C* **85** (2012) 045504 .
- [14] F. Šimkovic, A. Faessler, H. Mütter, V. Rodin, M. Stauf, *Phys. Rev. C* **79** (2009) 055501 .
- [15] H.V. Klapdor-Kleingrothaus, I.V. Krivosheina, *Mod. Phys. Lett. A* **21** (2006) 1547 .
- [16] F.T. Avignone, S.R. Elliott, J. Engel, *Rev. Mod. Phys.* **80** (2008) 481 .
- [17] M. Doi, T. Kotani, E. Takasugi, *Prog. Theor. Phys. (Suppl.)* **83** (1985) 1 .
- [18] F. Šimkovic, A. Faessler, V.A. Rodin, P. Vogel, J. Engel, *Phys. Rev. C* **77** (2008) 045503 .
- [19] R. Machleidt, *Phys. Rev. C* **63** (2001) 024001 .
- [20] S. Weinberg, *The Quantum Theory of Fields, v. I Foundations*, Cambridge University Press, 1995.
- [21] J.D. Bjorken, S.D. Drell, *Relativistic Quantum Mechanics, v. I*, McGraw-Hill, 1964.
- [22] B. Mukhopadhyay, personal communication, 2013.
- [23] P.D. Dobbie, B. Külebi, S.L. Casewell, M.R. Burleigh, Q.A. Parker, R. Baxter, K.A. Lawrie, S. Jordan, D. Koester, *MNRAS* **428** (2013) L16 .

## Appendix A: Calculations of traces

Here we provide the invariant functions  $A_i$  entering the positron-electron annihilation probability (3.4), resulting from calculations of traces and summing over the photon linear polarizations. The calculations are made in the Coulomb gauge, putting  $\epsilon^0 = \epsilon'^0 = 0$  and using

$$\sum_{\epsilon} \epsilon_i \epsilon_j = \delta_{ij} - \hat{\mathbf{k}}_i \hat{\mathbf{k}}_j, \quad \sum_{\epsilon'} \epsilon'_i \epsilon'_j = \delta_{ij} - \hat{\mathbf{k}}'_i \hat{\mathbf{k}}'_j.$$

$$m_e^4 A_0 = \frac{(k \cdot k')^2}{(p_f \cdot k)(p_f \cdot k')} - a_0, \quad (\text{A1})$$

$$m_e^4 A_1 = \{(-a_3 + a_5 + a_{11})(p_f \cdot k) + [(p_f \cdot k') - (k \cdot k') + a_1 - a_2]a_{13}\} / (p \cdot k)^2, \quad (\text{A2})$$

$$\begin{aligned} m_e^4 A_2 = & \{[a_1 - a_4 + a_{12} - a_{13} + (a_8 - a_6)/2](p_f \cdot k) + [a_3 - a_{10} - a_{11} + a_{12} \\ & + (a_6 - a_8)/2](p_f \cdot k') + [-a_1 - a_3 + a_4 + a_{11} - a_{12} + (a_5 + a_7)/2](k \cdot k') \\ & + (a_9 - a_3)a_{11} + (a_{13} - a_4)a_1 + a_3 a_{10}\} / (p_f \cdot k)(p_f \cdot k'), \end{aligned} \quad (\text{A3})$$

$$m_e^4 A_3 = \{[(p_f \cdot k) - (k \cdot k') + a_3 - a_9]a_{10} + (-a_1 + a_4 + a_7)(p_f \cdot k')\} / (p_f \cdot k')^2. \quad (\text{A4})$$

Further,

$$a_0 = 1 + (\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}')^2, \quad (\text{A5})$$

$$a_1 = (\mathbf{p}_f \cdot \mathbf{p}_{e+}) - (\mathbf{p}_f \cdot \hat{\mathbf{k}}')(\mathbf{p}_{e+} \cdot \hat{\mathbf{k}}'), \quad (\text{A6})$$

$$a_2 = (\mathbf{p}_{e+} \cdot \hat{\mathbf{k}}) - (\mathbf{p}_{e+} \cdot \hat{\mathbf{k}}')(\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}'), \quad (\text{A7})$$

$$a_3 = (\mathbf{p}_f \cdot \mathbf{p}_{e+}) - (\mathbf{p}_f \cdot \hat{\mathbf{k}})(\mathbf{p}_{e+} \cdot \hat{\mathbf{k}}), \quad (\text{A8})$$

$$a_4 = (\mathbf{p}_f \cdot \hat{\mathbf{k}}') - (\mathbf{p}_f \cdot \hat{\mathbf{k}})(\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}'), \quad (\text{A9})$$

$$a_5 = (\mathbf{p}_f \cdot \mathbf{p}_{e+}) - (\mathbf{p}_f \cdot \hat{\mathbf{k}})(\mathbf{p}_{e+} \cdot \hat{\mathbf{k}}) - (\mathbf{p}_f \cdot \hat{\mathbf{k}}')(\mathbf{p}_{e+} \cdot \hat{\mathbf{k}}') + (\mathbf{p}_f \cdot \hat{\mathbf{k}})(\mathbf{p}_{e+} \cdot \hat{\mathbf{k}}')(\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}'), \quad (\text{A10})$$

$$a_6 = (\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}') \left[ -(\mathbf{p}_f \cdot \hat{\mathbf{k}}') + (\mathbf{p}_f \cdot \hat{\mathbf{k}})(\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}') \right], \quad (\text{A11})$$

$$a_7 = (\mathbf{p}_f \cdot \mathbf{p}_{e+}) - (\mathbf{p}_f \cdot \hat{\mathbf{k}})(\mathbf{p}_{e+} \cdot \hat{\mathbf{k}}') - (\mathbf{p}_f \cdot \hat{\mathbf{k}}')(\mathbf{p}_{e+} \cdot \hat{\mathbf{k}}) + (\mathbf{p}_f \cdot \hat{\mathbf{k}})(\mathbf{p}_{e+} \cdot \hat{\mathbf{k}})(\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}'), \quad (\text{A12})$$

$$a_8 = (\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}') \left[ -(\mathbf{p}_f \cdot \hat{\mathbf{k}}) + (\mathbf{p}_f \cdot \hat{\mathbf{k}}')(\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}') \right], \quad (\text{A13})$$

$$a_9 = (\mathbf{p}_{e+} \cdot \hat{\mathbf{k}}') - (\mathbf{p}_{e+} \cdot \hat{\mathbf{k}})(\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}'), \quad (\text{A14})$$

$$a_{10} = \mathbf{p}_f^2 - (\mathbf{p}_f \cdot \hat{\mathbf{k}}')^2, \quad (\text{A15})$$

$$a_{11} = (\mathbf{p}_f \cdot \mathbf{k}) - (\mathbf{p}_f \cdot \hat{\mathbf{k}}')(\mathbf{k} \cdot \hat{\mathbf{k}}'), \quad (\text{A16})$$

$$a_{12} = \mathbf{p}_f^2 - (\mathbf{p}_f \cdot \hat{\mathbf{k}})^2 - (\mathbf{p}_f \cdot \hat{\mathbf{k}}')^2 + (\mathbf{p}_f \cdot \hat{\mathbf{k}})(\mathbf{p}_f \cdot \hat{\mathbf{k}}')(\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}'), \quad (\text{A17})$$

$$a_{13} = \mathbf{p}_f^2 - (\mathbf{p}_f \cdot \hat{\mathbf{k}})^2. \quad (\text{A18})$$

Here  $\hat{\mathbf{a}} = \mathbf{a}/|\mathbf{a}|$  is the unit vector. The invariant function  $A_0$  arises from the part of traces that do not contain the factors  $(\epsilon \cdot p_f)$  and  $(\epsilon' \cdot p_f)$ . If one puts  $\mathbf{p}_f = 0$ , one obtains the

positron-electron annihilation probability in the laboratory frame of reference. At threshold positron energies one gets for  $\Gamma_0$

$$\Gamma_0 = \frac{\alpha^2 \pi}{m_e^2}, \quad (\text{A19})$$

which provides for the annihilation cross section

$$\sigma = \frac{\alpha^2 \pi}{v m_e^2}, \quad (\text{A20})$$

where  $v$  is the positron velocity.